Multiplied configurations characterized by their closed substructures

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Abstract

We propose some new method of constructing configurations, which consists in consecutive inscribing copies of one underlying configuration. A uniform characterization of the obtained class and the one introduced in [5], which makes use of some covering by family of closed substructures, is given.
MSC 2000: 51D20, 51E26
Key words: partial linear space, duality, correlation, closed substructure, Shult axiom.

1 Introduction

In a series of papers (cf. [4], [5]) we have defined (and studied) two operations of "multiplying" partial linear spaces; the first operation can be applied to any partial linear space determined by a quasi difference set (with this set ditinguished), and the second one to any self dual structure (with its correlation distinguished). In both cases the resulting partial linear space $\mathfrak{N}$ can be covered by some family $\mathcal{B}$ of its closed substructures, each one isomorphic to the structure $\mathfrak{M}$ which was multiplied. In many examples this family $\mathcal{B}$ remains invariant under automorphisms of $\mathfrak{N}$; if so the automorphisms of $\mathfrak{N}$ can be easily determined.
There are some other operations which have this property. In section 2 we give an example of such a simple and natural operation, which does not coincide with the operations introduced in [5].

It turns out that some abstract, combinatorial properties of the covering $\mathcal{B}$ are sufficient to characterize the geometry of $\mathfrak{N}$. Roughly speaking, two observations are crucial. Let $\mathcal{B} = \{B_i \colon i \in I\}$.

> If $B_i = (B'_i, B''_i) \in \mathcal{B}$ then on every line $L \in B''_i$ there is exactly one point L^∞ of $\mathfrak{N}$ not in B'_i. All these "added" points are the points of some other closed substructure $B_j = (B'_j, B''_j) \in \mathcal{B}$.
>
> This leads to a map $\rho\colon i \longmapsto j$, and maps $B''_i \ni L \longmapsto L^\infty \in B'_{\rho(i)}$. The first one determines on I the structure of a cyclic group. The second are, in known examples, correlations.

In the paper we propose a system of conditions which express these properties in a more elementary language and we prove the representation theorem. Most of the previously investigated multiplied partial linear spaces satisfy this condition system, but the class of models satyfsying our conditions is even much wider.

2 Multyplying by dualisation

First, we recall that an incidence structure $\mathfrak{M} = \langle S, \mathcal{L}, \mathrel{\mathsf{I}} \rangle$ with $\mathrel{\mathsf{I}} \subset S \times \mathcal{L}$, is a *partial linear space* (PLS) provided that its every line is on at least two points, every point is on at least two lines and the following uniqueness condition is satisfied

$$\text{if } a, b \in S,\ k, m \in \mathcal{L},\ \text{and } a, b \mathrel{\mathsf{I}} k, m,\ \text{then } a = b \text{ or } k = m.$$

Let us adopt a convention that in $\mathfrak{M}$ sets S and $\mathcal{L}$ are disjoint. Now we introduce some notations we use further. We write $k = \overline{a, b}$ if $S \ni a, b \mathrel{\mathsf{I}} k \in \mathcal{L}$ and $a \neq b$; similarly $k \sqcap m = a$ means that $a \mathrel{\mathsf{I}} k, m$ and $k \neq m$. The phrase $k \sqcap m = \emptyset$ is used when there is no such a that $a \mathrel{\mathsf{I}} k, m$. If we write $a \sim b$ it means that there exists in $\mathfrak{M}$ a line k such that $a, b \mathrel{\mathsf{I}} k$. The degree of a point p in $\mathfrak{M}$ we denote by $r_p := |\{l \in \mathcal{L} : p \mathrel{\mathsf{I}} l\}|$, and dually the size of a line $r_l := |\{p \in S : p \mathrel{\mathsf{I}} l\}|$. We call the structure $\mathfrak{M}$ *Shultenian* (or Γ-*space*) (cf. [1]) if for every triangle (a, b, c) of $\mathfrak{M}$ and for every point d on the line $\overline{a, b}$ holds $c \sim d$.

Let us remind the construction of multyplying partial linear spaces using their correlations, considered in [5].

Let $\mathfrak{M}_0 = \langle \mathrm{S}_0, \mathrm{L}_0, \mathrel{\mathsf{I}}_0 \rangle$ be a partial linear space with a correlation $\varkappa_0$ and let $k > 2$ be an integer. We define $\mathfrak{M} = k \circledast_{\varkappa_0} \mathfrak{M}_0$ as follows. Let $M = C_k \times \mathrm{S}_0$, $\mathcal{L} = C_k \times \mathrm{L}_0$. We apply the following convention:

- (i, a) is a point, where $i \in C_k$ and $a \in S_0$;
- $[i, l]$ is a line, where $i \in C_k$ and $l \in \mathrm{L}_0$.

Then, the relation $\mathrel{\mathsf{I}}$ of incidence of $\mathfrak{M}$ is characterized by the condition

$$(i, a) \mathrel{\mathsf{I}} [j, l] \text{ iff, either } i = j \text{ and } a \mathrel{\mathsf{I}}_0 l, \text{ or } i = j + 1 \text{ and } a = \varkappa_0(l). \tag{1}$$

and we set $\mathfrak{M} = \langle M, \mathcal{L}, \mathrel{\mathsf{I}} \rangle$. The structure

$$k \circledast_{\varkappa_0} \mathfrak{M}_0 := \langle M, \mathcal{L}, \mathrel{\mathsf{I}} \rangle$$

will be referred to as *a correlative multiplying of* $\mathfrak{M}_0$.

In this section we adopt a dualisation (instead of correlation) as a convenient tool to multiply partial linear spaces. Let $\mathfrak{M}_0 = \langle \mathrm{S}_0, \mathrm{L}_0, \mathrel{\mathsf{I}}_0 \rangle$ be a partial linear space, and $\mathfrak{M}_0^\circ$ be the partial linear space dual to $\mathfrak{M}_0$. We

build a new structure $\mathfrak{M} = G \circledast_{\circ} \mathfrak{M}_0$, where G is a cyclic group of an even (greater than 2) or infinite rank, as follows. Let $i \in G$, we put

$$M_i = \begin{cases} \{i\} \times \mathrm{S}_0 & \text{for even } i \\ \{i\} \times \mathrm{L}_0 & \text{for odd } i, \end{cases}$$

$$\mathcal{L}_i = \begin{cases} \{i\} \times \mathrm{L}_0 & \text{for even } i \\ \{i\} \times \mathrm{S}_0 & \text{for odd } i. \end{cases}$$

Then we set $M := \bigcup_{i \in G} M_i$ and $\mathcal{L} := \bigcup_{i \in G} \mathcal{L}_i$. According to the terminology, where (i, a) is a point and $[i, b]$ is a line of $\mathfrak{M}$, we introduce the relation $\mathbf{I} \subset M \times \mathcal{L}$ as follows:

$$(i, a) \ \mathbf{I} \ [j, b] \text{ iff, either } i = j \text{ and } a \ \mathbf{I}_0 \ b \text{ (or } b \ \mathbf{I}_0 \ a), \text{ or } i = j + 1 \text{ and } a = b. \tag{2}$$

Finally, we put $\mathfrak{M} = \langle M, \mathcal{L}, \mathbf{I} \rangle$.
Obviously, the structure $G \circledast_{\circ} \mathfrak{M}_0$ is a PLS. Another immediate observation, based on the definition, is that $|M| = |\mathcal{L}| = \frac{|G|}{2} \cdot |\mathrm{S}_0| + \frac{|G|}{2} \cdot |\mathrm{L}_0|$. The following is evident.

Lemma 2.1. *Assume that $\mathfrak{M}_0$ is a PLS with constant point degree κ and line size ρ. Let a be a point or a line of $\mathfrak{M}_0$, $i \in G$ and $\mathfrak{M} = G \circledast_{\circ} \mathfrak{M}_0$.*

If (i, a) is a point of $\mathfrak{M}$ then the degree of (i, a) is	*If (i, a) is a line of $\mathfrak{M}$ then the size of (i, a) is*
$\kappa + 1$ *if i is even*	$\rho + 1$ *if i is even*
$\rho + 1$ *if i is odd.*	$\kappa + 1$ *if i is odd.*

Consequently, in $\mathfrak{M}$ there are $\frac{|G|}{2} \cdot |\mathrm{S}_0|$ points and the same number of lines with degree and size (respectively) $\kappa + 1$, and $\frac{|G|}{2} \cdot |\mathrm{L}_0|$ points and lines with degree and size $\rho + 1$.

The construction discussed in the paper has some advantages, when we compare it with the one considered in [5]. Firstly, it can be applied to any partial linear space, not necessarily admitting any correlation. Note, however, that consequently, we arrive to not necessarily (regular) configurations. The obtained structures are more artificial.

Example 2.2. Let $\mathfrak{M}_0$ be an incidence structure consisting of one line c and two points a, b on c, i.e. $\mathfrak{M}_0 = \langle \{a, b\}, \{c\}, \{a, b\} \times \{c\} \rangle$. Such structure we call a segment. Note, that point degree and line size in a segment are distinct. Let $G = C_6$ and consider $\mathfrak{M} = G \circledast_{\circ} \mathfrak{M}_0$ (see Figure 1). In $\mathfrak{M}$ we observe two classes of points and of lines. One of them, derived from $\mathfrak{M}_0$, contains points with degree 2 and lines with size 3. The second one, derived from $\mathfrak{M}_0^{\circ}$, contains points with degree 3 and lines with size 2. As we see, the obtained structure is not a regular configuration.

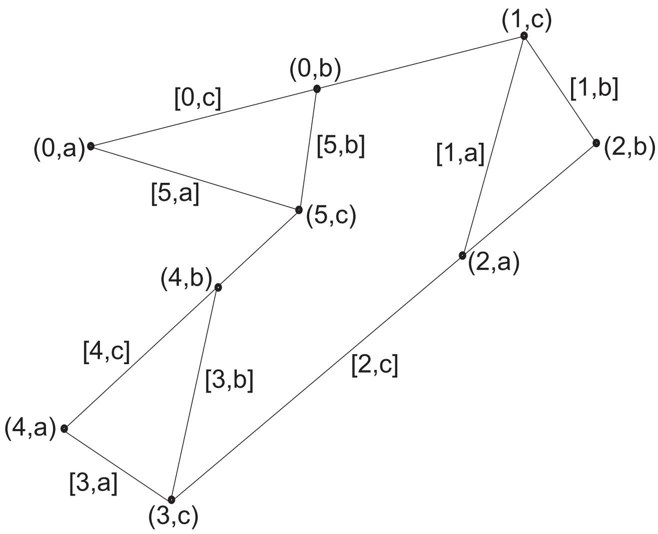

Fig. 1: The configuration $C_6 \circledast_\circ \mathfrak{M}_0$, where $\mathfrak{M}_0$ is a segment (cf. Example 2.2)

Proposition 2.3. *The map* $\varkappa = (\varkappa', \varkappa'')$, $\varkappa' \colon M \longrightarrow \mathcal{L}$, $\varkappa'' \colon \mathcal{L} \longrightarrow M$ *defined by the formula*

$$\varkappa' \left((i,x)\right) = [1-i, x], \quad \varkappa'' \left([i,y]\right) = (1-i, y)$$

is an involutive correlation of $\mathfrak{M} = G \circledast_\circ \mathfrak{M}_0 = \langle M, \mathcal{L}, \mathbf{I} \rangle$. *Consequently,* $\mathfrak{M}$ *is self dual.*

Proof. The map $\varkappa$ is a bijection of the structure $\mathfrak{M}$, which transforms the set of points onto the set of lines and dually. Consider (i,a) and $[j,b]$ such that $(i,a) \mathbf{I} [j,b]$. The images of them under the map $\varkappa$ are $[1-i,a]$ and $(1-j,b)$, respectively.

It is seen, that for $i = j$ the map $\varkappa$ preserves the relation $\mathbf{I}$ in $\mathfrak{M}$. In the case when $i = j+1$ we get $[1-i,a] = [-j,a]$ and $\varkappa'' ([j,b]) \mathbf{I} \varkappa' ((i,a))$ since in that case $a = b$. □

Proposition 2.4. *Let* $\mathfrak{M}_0^\circ$ *be dual to a partial linear space* $\mathfrak{M}_0$. *Then* $G \circledast_\circ \mathfrak{M}_0 \cong G \circledast_\circ \mathfrak{M}_0^\circ$.

Proof. Note that the map δ of the form $\delta((i,a)) = (i+1,a)$, where $(i,a) \in G \times \mathrm{S}_0$ or $(i,a) \in G \times \mathrm{L}_0$ transforms $G \circledast_\circ \mathfrak{M}_0$ onto $G \circledast_\circ \mathfrak{M}_0^\circ$, and it is a required isomorphism. □

Proposition 2.5. *Let* $\mathfrak{M} = C_k \circledast_\circ \mathfrak{M}_0$, *where* k *is even. If* $\mathfrak{M}_0$ *is a self dual structure with an involutive correlation* $\varkappa_0$ *then* $\mathfrak{M} \cong k \circledast_{\varkappa_0} \mathfrak{M}_0$.

Proof. Let us consider the following map $\delta : \mathfrak{M} \longrightarrow k \circledast_{\varkappa_0} \mathfrak{M}_0$:

$$\delta((i,x)) = \begin{cases} (i,x) & \text{for } i = 2t \\ (i, \varkappa_0\,(x)) & \text{for } i = 2t+1, \end{cases}$$

where $(i,x) \in C_k \times (\mathrm{S}_0 \cup \mathrm{L}_0)$. It is evident that δ is a bijection and it preserves the relation of incidence. Hence, this map is a required isomorphism. □

2.1 Covering by closed substructures

Now, we investigate some properties of $\mathfrak{M} = G \circledast_\circ \mathfrak{M}_0$ involving closed substructures. We are interested in similar results to that obtained in [5], but in our more general settings. A substructure $\langle B', B'' \rangle$ of $\langle M, \mathcal{L}, \mathrel{|} \rangle$ is a *closed substructure* if it satisfies the following two conditions:

1 $(\forall a_1, a_2 \in B')(\forall l \in \mathcal{L})[a_1, a_2 \mathrel{|} l \wedge a_1 \neq a_2 \implies l \in B'']$,

2 $(\forall l_1, l_2 \in B'')(\forall a \in M)[a \mathrel{|} l_1, l_2 \wedge l_1 \neq l_2 \implies a \in B']$.

Fact 2.6. *Let $\mathfrak{M}_0 = \langle \mathrm{S}_0, \mathrm{L}_0, \mathrel{|}_0 \rangle$ be a partial linear space and $i \in G$. Consider $\mathfrak{M} = G \circledast_\circ \mathfrak{M}_0 = \langle M, \mathcal{L}, \mathrel{|} \rangle$ and the map $\varepsilon_i = (\varepsilon_i', \varepsilon_i'')$ defined by*

$$\varepsilon_i' \colon \begin{cases} a \mapsto (i,a) \in M & \textit{for } i = 2t \\ a \mapsto [i,a] \in \mathcal{L} & \textit{for } i = 2t+1 \end{cases}$$

$$\varepsilon_i'' \colon \begin{cases} l \mapsto [i,l] \in \mathcal{L} & \textit{for } i = 2t \\ l \mapsto (i,l) \in M & \textit{for } i = 2t+1 \end{cases}$$

for $a \in \mathrm{S}_0$, $l \in \mathrm{L}_0$. The image $\mathrm{im}(\varepsilon_i)$ of $\mathfrak{M}_0$ under ε_i is a closed substructure of $\mathfrak{M}$ for every $i \in G$. It is isomorphic to $\mathfrak{M}_0$ or to $\mathfrak{M}_0^\circ$, as i is even or odd, respectively.

Our goal is to characterize closed substructures of $\mathfrak{M} = G \circledast_\circ \mathfrak{M}_0$ using terms of the geometry of $\mathfrak{M}$. In [5] we propose certain external definitions:

$$\text{if } p = (i,p'), q = (j,q') \in M, \text{ then } \quad p \diamond q \text{ iff } i = j \wedge (\exists l \in \mathcal{L})\big[p, q \mathrel{|} l\big], \quad (3)$$

$$\text{if } p = (i,p') \in M, l = [j,l'] \in \mathcal{L}, \text{ then } \quad p \wr l \text{ iff } p \mathrel{|} l \wedge i = j+1. \quad (4)$$

Consider above definitions in $\mathfrak{M}$. The formula

$$(\forall p, q \in M)\Big[p \diamond q \iff (\exists l \in \mathcal{L})\big[p, q \mathrel{|} l \ \wedge\ \neg(p \wr l) \ \wedge\ \neg(q \wr l)\big]\Big]. \quad (5)$$

express the relation $\diamond$ in the language of $(\mathfrak{M}, \wr)$ (cf. [5]). Assume that $\mathfrak{M}_0$ is connected. Let $\blacklozenge \subseteq M \times M$ be the transitive closure of the relation $\diamond$. Then, the closed substructure $\mathrm{im}(\varepsilon_i)$ of $\mathfrak{M}$ is the equivalence class of a point (i,a) of $\mathfrak{M}$ under the relation $\blacklozenge$.

The covering, mentioned in 2.6, can be easily recovered if the lines size and the points degree of the underlying structure $\mathfrak{M}_0$ are distinct.
We write $\mathrm{r}(x)$ for degree of x, if x is a point, or for size of x, if x is a line. Then, straightforward from 2.1 we get:

Corollary 2.7. *Let $\mathfrak{M}_0$ be a finite configuration with point degree κ and line size $\rho \neq \kappa$. Then for each point p and each line l of $G \circledast_\circ \mathfrak{M}_0$ we have*

$$p \wr l \text{ iff } p \mathrel{\mathbf{I}} l \wedge \mathrm{r}(p) = \mathrm{r}(l).$$

However, if the assumptions of 2.7 are not valid, in particular when $\mathfrak{M}_0$ is self dual, we must use more complex methods to restore the covering. For $p \in M$, $l \in \mathcal{L}$ we define the following relation:

$$p \wr_1 l \iff p \mathrel{\mathbf{I}} l \wedge (\exists q \in M)\Big[q \neq p \wedge q \mathrel{\mathbf{I}} l \wedge (\forall k, m, n \in \mathcal{L})\big[l \neq k, m, n \wedge m \neq n \wedge q \mathrel{\mathbf{I}} k \wedge p \mathrel{\mathbf{I}} m, n \wedge k \cap m \neq \emptyset \implies k \cap n = \emptyset\big]\Big]. \quad (6)$$

Proposition 2.8. *Let $G \circledast_\circ \mathfrak{M}_0 = \langle M, \mathcal{L}, \mathbf{I} \rangle$ and $\mathfrak{M}_0$ be Shultenian. Moreover, assume that $\mathfrak{M}_0$ has no 2-element lines, and any two collinear points of $\mathfrak{M}_0$ can be completed to a triangle in $\mathfrak{M}_0$ (equivalently: every line of $\mathfrak{M}_0$ is contained in a plane), and every two intersecting lines from $\mathfrak{M}_0$ determine a plane. Then $p \wr_1 l$ iff $p \wr l$ for every $p \in M$ and $l \in \mathcal{L}$.*

Proof. Let $\mathfrak{M} = G \circledast_\circ \mathfrak{M}_0 = \langle M, \mathcal{L}, \mathbf{I} \rangle$, where $\mathfrak{M}_0 = \langle \mathrm{S}_0, \mathrm{L}_0, \mathbf{I}_0 \rangle$, and $p \in M_i \subset M$, $l \in \mathcal{L}_j \subset \mathcal{L}$.

Let i be an even number from a cyclic group $G = C_k$ ($k > 2$) with even k, or from $G = \mathbb{Z}$. There are $p' \in \mathrm{S}_0$, $l' \in \mathrm{L}_0$ such that $p = (i, p')$ and $l = [j, l']$. Let us assume that $p \wr l$, in other words $p \mathrel{\mathbf{I}} l$ and $i = j + 1$. Take a point $q \neq p$ and a line $k \neq l$ such that $q \mathrel{\mathbf{I}} l, k$. Next, consider a line m such that $m \sqcap l = p$, $m \sqcap k = r$. Observe $\mathfrak{M}_{(p)}$ (see fig. 2). Note that neighbourhoods of a point in the structure $\mathfrak{M}$ and in $k \circledast_\varkappa \mathfrak{M}_0$, considered in [5] (where $\varkappa$ is the involutory correlation of $\mathfrak{M}_0$) are isomorphic both for $k = 3$ and for an arbitrary $k > 3$. Therefore, the line m is the unique one in $\mathfrak{M}_{(p)}$, which crosses any line passing through q, it follows from [5, Lemma 2.4]. Consequently, $p \wr_1 l$ holds.

Now, assume that $p \wr l$ does not hold. For p not laying on l we get $\neg(p \wr_1 l)$ immediately. Let $\neg(p \wr l)$ and $p \mathrel{\mathbf{I}} l$ hold, so $i = j$. Consider arbitrary point $q = (i, q') \neq p$ such that $q \mathrel{\mathbf{I}} l$. From assumptions there exists a triangle (q', p', r') in $\mathfrak{M}_0$ and a point $s' \neq p', r'$ on $\overline{q', r'}$. Set $k = [i, \overline{q', r'}]$, $m = [i, \overline{p', r'}]$, and $n = [i, \overline{p', s'}]$, which existence follows by the Shult axiom. Next, let $q = (i + 1, l') \mathrel{\mathbf{I}} l$ and $k = [i + 1, p']$, so $q \mathrel{\mathbf{I}} k$. Take any two lines $m, n \neq l$ of the form $[i, x']$ passing through p; then k crosses them both (see details in [5]). This proves that $p \wr_1 l$ does not hold.

If $i \in G$ is odd then for $p = (i, p')$, $l = [j, l']$ we have $p' \in \mathrm{L}_0$, $l' \in \mathrm{S}_0$. If we assume that $p \wr l$ then directly from analysing in $G \circledast_\circ \mathfrak{M}_0$ the neighbourhood of the point p we obtain that $p \wr_1 l$.

In order to close our proof we consider the case with i odd and $\neg(p \wr l)$. Let $q' \in \mathrm{L}_0$ be a line such that q' intersects p'. Every two intersecting lines determine a plane in $\mathfrak{M}_0$. Hence, we get the existance of a line $r' \in \mathrm{L}_0$

such that q', p', r' are sides of one triangle. Now, the claim is provided by arguments analogous to those used for i even. $\square$

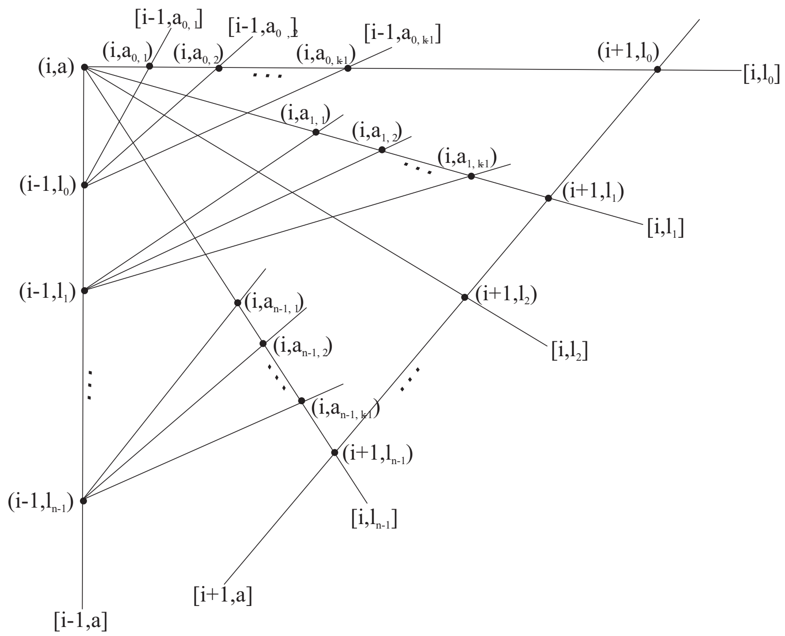

Fig. 2: $\mathfrak{M}_{((i,a))}$ – the neighbourhood of a point (i,a) in $\mathfrak{M} = G \circledast_\circ \mathfrak{M}_0$

Let us introduce another two relations $\lhd, \wr_2 \subseteq M \times \mathcal{L}$ given by the formulas:

$$a \lhd l \iff \neg(a \mathbin{\mathbf{I}} l) \wedge (\exists!\, m)\, [a \mathbin{\mathbf{I}} m \wedge l \sqcap m = \emptyset]; \tag{7}$$

$$p \wr_2 l \iff (\exists a \in M)[l \mathbin{\mathbf{I}} p \not\sim a \lhd l]. \tag{8}$$

Proposition 2.9. *Let* $G \circledast_\circ \mathfrak{M}_0 = \langle M, \mathcal{L}, \mathbf{I}\rangle$ *and* $\mathfrak{M}_0 = \langle \mathrm{S}_0, \mathrm{L}_0, \mathbf{I}_0\rangle$ *be a partial linear space with constant size* > 2 *of each line. Assume, that for every pair* $(a,l) \in \mathrm{S}_0 \times \mathrm{L}_0$ *such that* a *is outside* l *there is a line through* a*, which misses* l *and there is a point on* l*, which is not collinear with* a*. Then* $p \wr_2 l$ *iff* $p \wr l$ *for every* $p \in M$ *and* $l \in \mathcal{L}$

Proof. Let $\mathfrak{M} = G \circledast_\circ \mathfrak{M}_0 = \langle M, \mathcal{L}, \mathbf{I}\rangle$.

Let us take a point $p = (i, p') \in M$ and a line $l = [j, l'] \in \mathcal{L}$ such that $p \wr l$. Then $p \mathbin{\mathbf{I}} l$ and $i = j+1$. Let us consider the neighbourhood of a point a such that $a \lhd l$. Since $a \lhd l$, then $\mathrm{r}(a) - 1$ is the size of l in $\mathfrak{M}_{(a)}$. From assumptions, for every line $m = [j-1, m'] \in \mathcal{L}$, which does not pass through a, there is a line from $\mathrm{im}(\varepsilon''_{j-1})$ through a, which misses m. Thus, we have at least two lines through a, which do not cross m in $\mathfrak{M}_{(a)}$. In $\mathfrak{M}_{(a)}$ size of

every line from $\mathrm{im}(\varepsilon''_{j-2})$, which does not pass through a, equals 2. Hence, in $\mathfrak{M}_{(a)}$ size of every line distinct from l, which does not pass through a is less than $\mathrm{r}(a)-1$. Therefore, the line l is uniquely determined by its relation to a. The point p is also the unique one on the line l, which is not collinear with a. Consequently, we conclude that $p \wr_2 l$.

The converse implication can be proved by applying the converse reasoning. $\square$

Propositions 2.7, 2.8 and 2.9 together with 2.6 yield the following:

Corollary 2.10. *Let $\mathfrak{M}_0$ be connected. Under assumptions of each of 2.7, 2.8 and 2.9 the covering of $G \circledast_\circ \mathfrak{M}_0$ by the family of closed substructures $\{\mathrm{im}(\varepsilon_i) : i \in G\}$ is definable in $G \circledast_\circ \mathfrak{M}_0$. Consequently it is preserved by all automorphisms of $G \circledast_\circ \mathfrak{M}_0$.*

There are some significant examples of structures $G \circledast_\circ \mathfrak{M}_0$, of which automorphisms preserve their closed substructures. Proposition 2.7 provides a wide class of such examples. For instance we can adopt any finite affine, or slit space (cf. [2], [3]) as a structure $\mathfrak{M}_0$ we start from. Every projective plane satisfies assumptions of 2.8. Thus, the family of $G \circledast_\circ \mathfrak{M}_0$, where $\mathfrak{M}_0$ is a projective plane, is the next class of examples.

Let X be an arbitrary set. We write $\wp_m(X)$ for the set of m-element subsets of X. Let m be an integer with $1 \leq m < |X|$. Then $\mathbf{G}_m(X)$ is the incidence structure $\mathbf{G}_m(X) = \langle \wp_m(X), \wp_{m+1}(X), \subset \rangle$. We call this structure *combinatorial Grassmannian.* It is worth to note that $\mathbf{G}_2(5) = \mathfrak{D}$ is the Desargues configuration.

Fact 2.11. *If $m \neq n-1, n-2$, then the structure $\mathbf{G}_m(X)$ satisfies assumptions of 2.9.*

Proof. Consider a point a and a line L of $\mathbf{G}_m(X)$, such that a is outside L. If $a \cap L = \emptyset$ then we set $M := a \cup \{x\}$ with $x \in L$, and $b := L \setminus \{x\}$.
Let $a \cap L \neq \emptyset$. From assumptions there exists x such that $x \notin a \cup L$. Then we set $M := a \cup \{x\}$. There exist two $y, y' \in L$ such that $y, y' \notin a$, since $|a \cap L| \leq m-1$. Take $b \subset L$ with $y, y' \in b$.
In both above cases $|M \cap L| < m$ and $|b \cup a| > m+1$. $\square$

That way we obtain the second class of examples – structures $G \circledast_\circ \mathbf{G}_m(X)$.

Let $\mathfrak{F} = \langle F, +, \cdot, 0, 1 \rangle$ be a finite field, $\mathfrak{F} = GF(q)$. Assume that $2 \nmid q$. We construct the *Havlicek-Tietze configuration $HT(q)$* as follows:

- Its point universe is $X = F \times F$, with elements of X written as (a, b), $a, b \in F$.
- Its blocks are pairs $[\alpha, \beta]$ with $\alpha, \beta \in F$, we write $\mathcal{G}$ for the set of such blocks.

– The incidence relation is defined by $(a,b) \mathrel{\mathsf{I}} [\alpha,\beta]$ iff $a \cdot \alpha = b + \beta$.

Then $HT(q) := \langle X, \mathcal{G}, \mathsf{I} \rangle$. Note that $HT(3)$ is the Pappus configuration. In [5] we proved the following:

Fact 2.12. *Let $\mathfrak{A}$ be an affine plane over $\mathfrak{F}$ and $\mathcal{D}$ be the direction of a line l of $\mathfrak{A}$. Then $HT(q)$ results from $\mathfrak{A}$ by deleting the lines in $\mathcal{D}$. In particular, $HT(q)$ is a partial linear space with parallelism. Conversely, the family $\{c\colon a = c$ or $a = b$ or c is not collinear with $a, b\}$, where a, b are not collinear points of $HT(q)$, is the set $\mathcal{D}$.*

Now, it is seen that the assumptions of 2.9 hold for $HT(q)$. Hence, another set of examples in question is $G \circledast_\circ HT(q)$.

3 Synthetic characterization

Let $B = \{B_i = \langle B_i', B_i'' \rangle : i \in I\}$ be a family of connected closed substructures of a connected partial linear space $\mathfrak{M} = \langle M, \mathcal{L}, \mathsf{I} \rangle$ such that $\bigcup_{i\in I} B_i' = M$, $\bigcup_{i\in I} B_i'' = \mathcal{L}$. Let us introduce the following conditions

(1) $(\forall i_1, i_2 \in I)[B_{i_1}' \cap B_{i_2}' \neq \emptyset \implies B_{i_1}' = B_{i_2}']$

(2) $(\forall i_1, i_2 \in I)[B_{i_1}'' \cap B_{i_2}'' \neq \emptyset \implies B_{i_1}'' = B_{i_2}'']$

(3) $(\forall d \in B_i')(\exists m)[d \mathrel{\mathsf{I}} m \wedge m \notin B_i'']$

(4) $(\forall m \in B_i'')(\exists d)[d \mathrel{\mathsf{I}} m \wedge d \notin B_i']$

(5) $(\forall i \in I)[\langle M \setminus B_i', \mathcal{L} \setminus B_i'' \rangle$ is a closed substructure of $\langle M, \mathcal{L} \rangle]$

(6) $(\forall m_1, m_2, m_3)\big[p \mathrel{\mathsf{I}} m_1, m_2, m_3 \wedge d_1 \mathrel{\mathsf{I}} m_1 \wedge d_2 \mathrel{\mathsf{I}} m_2 \wedge d_3 \mathrel{\mathsf{I}} m_3 \wedge d_1, d_2, d_3 \notin B_i' \implies (\exists n)[d_1, d_2, d_3 \mathrel{\mathsf{I}} n \vee m_1 \notin B_i'' \vee m_2 \notin B_i'' \vee m_3 \notin B_i'']\big]$

(7) $(\forall d_1, d_2, d_3)\big[d_1, d_2, d_3 \mathrel{\mathsf{I}} n \wedge d_1 \mathrel{\mathsf{I}} m_1 \wedge d_2 \mathrel{\mathsf{I}} m_2 \wedge d_3 \mathrel{\mathsf{I}} m_3 \wedge m_1, m_2, m_3 \notin B_i'' \implies (\exists p)[p \mathrel{\mathsf{I}} m_1, m_2, m_3 \vee d_1 \notin B_i' \vee d_2 \notin B_i' \vee d_3 \notin B_i']\big]$

Note, that conditions in the following pairs: (1) and (2), (3) and (4), (6) and (7), are mutually dual.
Now, let us define a new relation $\rho \subseteq I \times I$, namely:

$$j \mathrel{\rho} i \iff (\exists l \in B_j'')(\exists a \in B_i')[a \mathrel{\mathsf{I}} l \wedge j \neq i]. \tag{9}$$

Then, as an immediate consequence of conditions (3) and (4), we obtain

Corollary 3.1. *For every $i \in I$ there exists $j \in I$ and for every $j \in I$ there exists $i \in I$ such that $i \mathrel{\rho} j$.*

Let us pay some more attention to the properties of B, which follow directly from the introduced conditions.

Lemma 3.2. *The following statement holds:*

$(\forall m_1, m_2)\big[p \mathrel{I} m_1, m_2 \wedge d_1 \mathrel{I} m_1 \wedge d_2 \mathrel{I} m_2 \wedge m_1 \in B''_i \wedge m_2 \in B''_i \wedge d_1, d_2 \notin B'_i \implies (\exists n)[d_1, d_2 \mathrel{I} n]\big]$.

Furthermore, the dual version of this statement is also valid.

Proof. It suffices to adopt $m_2 = m_3$ and $d_2 = d_3$ in conditions (6), (7). □

Lemma 3.3. *Let $d_1 \mathrel{I} m_1$, $d_2 \mathrel{I} m_2$, $m_1 \sqcap m_2 \neq \emptyset$ and $m_1, m_2 \in B''_j$. If $d_1 \in B'_i$ for some $i \neq j$ and $d_2 \notin B'_j$, then $d_2 \in B'_i$. Moreover, the dual version of this statement is true as well.*

Proof. Let us take some $I \ni i_1, i_2 \neq i$. Assume that $d_1 \in B'_{i_1}$, $d_2 \in B'_{i_2}$, $i_1 \neq i_2$, and $d_1 \mathrel{I} m_1 \in B''_j$, $d_2 \mathrel{I} m_2 \in B''_j$ and $m_1 \sqcap m_2 \neq \emptyset$. There exists a line n which joins d_1, d_2, from 3.2. Then, condition (5) yields that $n \notin B''_j$. Note, that $m_1, m_2 \notin B''_{i_1}, B''_{i_2}$. So, if moreover $n \notin B''_{i_1}$ or $n \notin B''_{i_2}$ then $d_1 \notin B'_{i_1}$ or $d_2 \notin B'_{i_2}$ follows from condition (5). Hence, $n \in B''_{i_1}$ and $n \in B''_{i_2}$ holds. Together with condition (2) it implies that $i_1 = i_2$, which closes our proof. □

Note, that since each of close substructure of $\mathfrak{M}$ is connected the conclusion of 3.3 remain also valid for $m_1 \sqcap m_2 = \emptyset$. Then, using 3.1 and 3.3, we obtain

Corollary 3.4. *The map ρ is a bijection of the set I.*

Directly from definitions we have

Fact 3.5. *If $a_j \in B'_{i_j}$, $j = 1, 2$ and $a_1 \sim a_2$ then $i_1 = i_2$, $i_1 \,\rho\, i_2$ or $i_2 \,\rho\, i_1$.*

Proposition 3.6. *For arbitrarily fixed $i_0 \in I$ and the map ρ defined in* (9) *the following holds:*

$$\langle\rho\rangle[i_0] = \{\rho^s\ (i_0) : s = 0, \pm 1, \pm 2, \ldots\} = I.$$

Then, the map ρ determines on I the structure of a group C_k (with $k = |I|$) for a finite set I or the structure of $\mathbb{Z}$ otherwise. Moreover, $i \,\rho\, j$ iff $j = i+1$.

Proof. The set $\langle\rho\rangle$ is a group of bijections, from 3.4. Let us fix some $i_0 \in I$. Then, $\langle\rho\rangle[i_0]$ is an orbit of the group $\langle\rho\rangle$. The equality $\langle\rho\rangle[i_0] = I$ is immediate from 3.5 and connectedness of $\mathfrak{M}$.

Obviously $\langle\rho\rangle$ is a cyclic group. Then, either $\langle\rho\rangle \cong C_k$ if k is a finite rank of $\langle\rho\rangle$ or $\langle\rho\rangle \cong \mathbb{Z}$ otherwise. We denote the rank of the group $\langle\rho\rangle$ as $r_{\langle\rho\rangle}$. For $s \in C_k$ (or $s \in \mathbb{Z}$) let us consider the map

$$s \overset{\gamma}{\longmapsto} \rho^s\ (i_0).$$

One can note that γ induces the following map $\langle\rho\rangle \ni \rho^s \overset{\rho}{\longmapsto} {}^s(i_0) \in I$.

We claim that γ is a bijection. Let us assume that $\rho^{s_2}\ (i_0) = \rho^{s_1}\ (i_0)$. It yields $\rho^{s_2 - s_1}\ (i_0) = i_0$ for some s_1, s_2 such that $r_{\langle\rho\rangle} > s_2 > s_1$. Therefore $\rho^k\ (i_0) = i_0$ holds for $k < r_{\langle\rho\rangle}$. If $I_1 := \{\rho^j\ (i_0) : j = 0, \ldots, k-1\}$ then $\rho^s\ (I_1) = I_1$ holds for any $s \in \mathbb{Z}$. Thus, $\langle\rho\rangle[i_0] \subset I_1$ and consequently $I = I_1$. It implies that $\rho^k\ (i) = i$ for any $i \in I$. Finally, we obtain that the rank of $\langle s\rangle$ is less than k, which contradicts our previous assumptions.

Let $s' = s + 1$ hold. From 3.4 $\rho\ (\rho^s\ (i_0)) = \rho^{s+1}\ (i_0)$ and i, $\rho\ (i)$ are ρ-related for any $i \in I$. Hence, $\gamma(s)\ \rho\ \gamma(s')$ holds. This proves that γ transfers the structure of the group C_k (or $\mathbb{Z}$) on $\langle\rho\rangle[i_0]$. □

Lemma 3.7. *Let $m_1, m_2, m_3 \in B''_j$ and $d_1, d_2, d_3 \in B'_i$ such that $d_1 \mid m_1$, $d_2 \mid m_2$, $d_3 \mid m_3$ for some $j, i \in I$, $j \neq i$. Then, m_1, m_2, m_3 meet in a point distinct from d_1, d_2, d_3 iff d_1, d_2, d_3 are on a line distinct from m_1, m_2, m_3.*

Proof. Let $p \neq d_1, d_2, d_3 \in B'_i$ be the common point of the lines $m_1, m_2, m_3 \in B''_j$. The condition (1) yields $d_1, d_2, d_3 \notin B'_j$, since $j \neq i$. Then, directly from condition (6) we get the existence of the line $n \neq m_1, m_2, m_3$, which connects d_1, d_2, d_3.

Analogously, based on conditions (2) and (7), we prove the converse implication. □

Now, we define another relation $\wr \subseteq M \times \mathcal{L}$ by the formula

$$a \wr m \text{ iff } a \mid m \text{ and } \neg(\exists i \in I)[a \in B'_i \wedge m \in B''_i] \tag{10}$$

Lemma 3.8. *The relation $\wr \subseteq M \times \mathcal{L}$ defined in* (10) *is a bijection acting from M onto $\mathcal{L}$.*

Proof. From condition (3) we get that for every $a \in M$ there exists such $m \in \mathcal{L}$ that $a \wr m$. Assume that there exist $a \in M$ and $m_1, m_2 \in \mathcal{L}$ such that $a \wr m_1$ and $a \wr m_2$. Then, from condition (5), $m_1 = m_2$. Thus, relation $\wr$ is a function. Conditions (4) and (5) yield that $\wr$ is a surjection and an injection, respectively. □

Next, let us introduce the following pair of transformations:

$$\begin{cases} \varkappa'\ (a) = m \text{ iff } a \wr m \\ \varkappa''\ (m) = a \text{ iff } a \wr m, \end{cases} \tag{11}$$

which are in fact functions, what follows from 3.8. Take $j, i \in I$ such that $j\ \rho\ i$. Then we have $\varkappa'\ (B'_j) = B''_i$ and $\varkappa''\ (B''_i) = B'_j$. To get a proper, full correlation acting from the structure B_j to B_i let us note the following.

Fact 3.9. *Let $j\ \rho\ i$ for $j, i \in I$ and $\varkappa = (\varkappa', \varkappa'')$ be the function introduced in* (11), *and $\phi'_j := \varkappa' \restriction B'_j$, $\psi''_i := \varkappa'' \restriction B''_i$. There exist maps $\phi''_j\colon B''_j \longrightarrow B'_i$, $\psi'_i\colon B'_i \longrightarrow B''_j$, which complete ϕ'_j and ψ''_i to the correlations*

$$\phi_j = (\phi'_j, \phi''_j), \quad \psi_i = (\psi'_i, \psi''_i),$$

where $\phi_j\colon B_j \longrightarrow B_i$, $\psi_i\colon B_i \longrightarrow B_j$, and $\phi''_j = (\psi'_i)^{-1}$, $\psi''_i = (\phi'_j)^{-1}$ hold.

Proof. The existence of maps $\phi''_j : B''_j \longrightarrow B'_i$ and $\psi'_i : B'_i \longrightarrow B''_j$ is immediate by 3.7. If $j\ \rho\ i$ then the transformation ψ'_i is the inverse of ϕ''_j, and ϕ'_j is the inverse of ψ''_i. □

4 Representation theorems

Now we present, in some sense, more general construction. We set $I = C_k$ or $I = \mathbb{Z}$. Let us consider a family of connected partial linear spaces $(\mathfrak{M}_i)_{i\in I} = \langle M_i, \mathcal{L}_i, \mathrm{I}_i\rangle$ and a family $(\phi_i)_{i\in I}$, where $\phi_i = (\phi'_i, \phi''_i)$ is a correlation such that $\phi_i \colon \mathfrak{M}_i \longrightarrow \mathfrak{M}_{i+1}$. Then we put

$$X_i := \{i\} \times M_i, \quad X = \bigcup_{i\in I} X_i, \qquad H_i := \{i\} \times \mathcal{L}_i, \quad H = \bigcup_{i\in I} H_i,$$

and we introduce the following relation I

$$(i,a)\ \mathrm{I}\ [j,m] \text{ iff, either } i = j \text{ and } a\ \mathrm{I}_i\ m, \text{ or } i = j+1 \text{ and } a = \phi''_j(m). \tag{12}$$

As a result we obtain the structure

$$\circledast_{i\in I}\,(\mathfrak{M}_i, \phi_i) := \langle X, H, \mathrm{I}\rangle. \tag{13}$$

Clearly $\mathfrak{B}_i = \langle X_i, H_i\rangle$ is a closed substructure of $\mathfrak{M} = \circledast_{i\in I}(\mathfrak{M}_i, \phi_i)$ isomorphic to $\mathfrak{M}_i$, for each $i \in I$, and $\mathfrak{M}$ is a partial linear space. As an another straightforward consequence of the definition we get

Fact 4.1. *The family $\{\mathfrak{B}_i : i \in I\}$ of closed substructures of the structure $\mathfrak{M}$ is a covering of $\mathfrak{M}$. This family satisfies all conditions from* (1) *to* (7)*.*

Proposition 4.2. *Let $\mathfrak{M}_i = \langle M_i, \mathcal{L}_i, \mathrm{I}_i\rangle$ be a connected partial linear space for each of $i \in I$ and $(\xi_i)_{i\in I} = (\xi'_i, \xi''_i)$ be a correlation of $\mathfrak{M}_i$ onto $\mathfrak{M}_{-i}$. Next, let $\mathfrak{M} = \circledast_{i\in I}(\mathfrak{M}_i, \phi_i) = \langle X, H, \mathrm{I}\rangle$ be the structure defined by* (13)*. Assume that*

$$\phi''_{-i}\xi'_i\phi''_{i-1} = \xi''_i.$$

Then, the structure $\mathfrak{M}$ is self dual and the map $\varkappa = (\varkappa', \varkappa'')$, $\varkappa' \colon X \longrightarrow H$, $\varkappa'' \colon H \longrightarrow X$, where

$$\varkappa'\ ((i,x)) = [-i, \xi'_i(x)], \quad \varkappa''\ ([i,y]) = (-i, \xi''_i(y))$$

is a correlation of $\mathfrak{M}$.

Proof. The map $\varkappa$ is a bijection of the structure $\mathfrak{M}$, which transforms the set of points onto the set of lines and dually. Let us take such (i,a) and $[j,m]$, that $(i,a)\ \mathrm{I}\ [j,m]$. Their images are $\varkappa'\ ((i,a)) = [-i, \xi'_i(a)] := [a']$ and $\varkappa''\ ([j,m]) = (-j, \xi''_i(m)) := (m')$.

Let us assume that $i = j$ and $a\ \mathrm{I}_i\ m$. Then, $(m')\ \mathrm{I}\ [a']$ as the map ξ_i is a correlation transforming $\mathfrak{M}_i$ onto $\mathfrak{M}_{-i}$.

Now, assume $i = j+1$ and $a = \phi''_j(m)$. So, $(m')\ \mathrm{I}\ [a']$ if $\phi''_{-i}(\xi'_i(a)) = \xi''_i(m)$. And next, we get $\phi''_{-i}(\xi'_i(\phi''_{i-1}(m))) = \xi''_i(m)$, which is the required condition. □

Theorem 4.3. *Let $\mathfrak{M} = \langle M, \mathcal{L}, \mathrm{I} \rangle$ be a connected partial linear space covered by a family of closed substructures $\{B_i = \langle B_i', B_i'' \rangle : i \in I\}$ satisfying all conditions from* (1) *to* (7)*. Let ρ be relation introduced in* (9)*, and $\phi_i = (\phi_i', \phi_i'')$ be the correlation, defined in 3.9, mapping B_i onto $B_{\rho(i)}$. Then $\mathfrak{M} \cong \circledast_{i\in I}(B_i, \phi_i)$.*

Proof. Let us define a map $\delta \colon \mathfrak{M} \longrightarrow \circledast_{i\in I}(B_i, \phi_i)$ by the formula:

$$\begin{array}{ll} \delta' \colon a \mapsto (i, a) & \text{for } a \in B_i' \\ \delta'' \colon l \mapsto [i, l] & \text{for } l \in B_i'' \end{array}, \text{ and we put } \delta = (\delta', \delta'').$$

Since the family $\{B_i = \langle B_i', B_i'' \rangle : i \in I\}$ covers $\mathfrak{M}$, every $a \in M$ and every $l \in \mathcal{L}$ has an image under δ. From conditions (1) and (2) this image is uniquely determined. Then the map δ is a function. Directly from the definition it is a surjection and an injection.
Assume that $a \mathrel{\mathrm{I}} l$ in $\mathfrak{M}$. Let $a \in B_i'$, $l \in B_i''$. Then $\delta'(a) = (i, a)$, $\delta''(l) = [i, l]$, and $\delta'(a) \mathrel{\mathrm{I}} \delta''(l)$ follows by (12).
Let $a \in B_i'$, $l \in B_j''$ and $i \neq j$. From (9) we get $j \mathrel{\rho} i$. Thus, 3.5 yields $i = j + 1$, and $\delta'(a) = (i, a) \mathrel{\mathrm{I}} [j, l] = \delta''(l)$ follows by (12). □

Note, that a family $\{\delta(B_i) \colon i \in I\}$ is a covering of $\circledast_{i\in I}(B_i, \phi_i)$ by a family of its closed substructures. For $a \in B_i'$, $l \in B_i''$ let us consider another map $\phi_i^\star = (\phi_i^{\star\prime}, \phi_i^{\star\prime\prime})$ such that

$$\phi_i^{\star\prime} \colon (i, a) \mapsto [i+1, \phi_i'(a)], \quad \text{and} \quad \phi_i^{\star\prime\prime} \colon [i, l] \mapsto (i+1, \phi_i''(l)).$$

The transformation $\phi_i^\star$ is a correlation, induced by ϕ_i, mapping $\delta(B_i)$ onto $\delta(B_{i+1})$.

Proposition 4.4. *Let $\circledast_{i\in I}(\mathfrak{M}_i, \phi_i)$ be the structure defined by* (13) *and $\mathfrak{M}_0 = \langle \mathrm{S}_0, \mathrm{L}_0, \mathrm{I}_0 \rangle$ be a PLS.*

(i) *If $I = \mathbb{Z}$ then $\circledast_{i\in I}(\mathfrak{M}_i, \phi_i) \cong \mathbb{Z} \circledast_\circ \mathfrak{M}_0$.*

(ii) *If $I = C_k$ with k even and $\phi_{k-1} \dots \phi_1 \phi_0 = \mathrm{id}$ then $\circledast_{i\in I}(\mathfrak{M}_i, \phi_i) \cong C_k \circledast_\circ \mathfrak{M}_0$.*

Proof. Let $I = \mathbb{Z}$, $\circledast_{i\in I}(\mathfrak{M}_i, \phi_i) = \langle X, H, \mathrm{I} \rangle$, and

$$\alpha_i := \phi_0^{-1} \dots \phi_{i-2}^{-1} \phi_{i-1}^{-1}, \quad \beta_i := \phi_{-1} \dots \phi_{i+1} \phi_i.$$

We introduce the following bijective map $\delta \colon \circledast_{i\in I} (\mathfrak{M}_i, \phi_i) \longrightarrow \mathbb{Z} \circledast_\circ \mathfrak{M}_0$ by the condition:

$$\delta((i, x)) = \begin{cases} (i, \alpha_i(x)) & \text{for } i > 0 \\ (i, x) & \text{for } i = 0 \\ (i, \beta_i(x)) & \text{for } i < 0 \end{cases}, \quad (i, x) \in X \cup H.$$

Let $(i,a) \mathrel{I} [j,m]$ for some $(i,a) \in X$, $[j,m] \in H$. First, we assume that $i = j$ and $a \mathrel{I}_i m$. Then $\alpha_i(a) \mathrel{I}_0 \alpha_i(m)$ (or $\alpha_i(m) \mathrel{I}_0 \alpha_i(a)$) and $\beta_i(a) \mathrel{I}_0 \beta_i(m)$ (or $\beta_i(m) \mathrel{I}_0 \beta_i(a)$) is evident.
Now let $i = j+1$ and $a = \phi''_j(m) = \phi''_{i-1}(m)$ and $i > 0$. If $j \neq 0$ then we obtain $\delta((i,a)) = (i, \alpha_i(a)) = (i, \alpha_{i-1}\phi^{-1}_{i-1}(a)) = (i, \alpha_{i-1}(m))$, and $\delta((j,m)) = \delta((i-1,m)) = (i-1, \alpha_{i-1}(m))$. If $j = 0$ then $a = \phi''_0(m)$ and we get $\delta((i,a)) = \delta((1,a)) = (1, \phi_0^{-1}(a)) = (1,m)$, $\delta((j,m)) = \delta((0,m)) = (0,m)$. We obtain the same result for $i \in \mathbb{Z}$ such that $i < 0$. If $i = 0$ then $a = \phi''_{-1}(m)$, and $\delta((i,a)) = (0,a)$, $\delta((j,m)) = \delta((-1, \phi''_{-1}(m))) = (-1,a)$.
Each case considered above yields $\delta((i,a)) \mathrel{I} \delta((j,m))$, follows by (1). Consequently, the map δ is a required isomorphism.

Now let $I = C_k$ with k even. Consider the map

$$\sigma((i,x)) = \begin{cases} (i, \alpha_i(x)) & \text{for } i \neq 0 \\ (i,x) & \text{for } i = 0 \end{cases}, \quad (i,x) \in X \cup H.$$

Obviously this map is a bijection acting from $\circledast_{i\in I}(\mathfrak{M}_i, \phi_i)$ onto $C_k \circledast_\circ \mathfrak{M}_0$. Let us check whether σ preserves the relation of incidence. The only not evident (not analogous to the case with $I = \mathbb{Z}$) case we have to consider is $(0,a) \mathrel{I} [k-1,m]$. Then $a = \phi''_{k-1}(m)$, $\sigma((0,a)) = (0,a)$ and $\sigma((k-1,m)) = (k-1, \alpha_{k-1}(m))$. Assumption $\phi_{k-1}\ldots\phi_1\phi_0 = \mathrm{id}$ yields $\phi_{k-1} = \phi_0^{-1}\ldots\phi_{k-2}^{-1} = \alpha_{k-1}$. Therefore, $\sigma((k-1,m)) = (k-1, \phi''_{k-1}(m)) = (k-1,a)$ and $\sigma((0,a)) \mathrel{I} \sigma((k-1,m))$ holds by (1). □

Proposition 4.5. *Let $\circledast_{i\in I}(\mathfrak{M}_i, \phi_i)$ be the structure defined by* (13). *If $I = C_k$ with odd k and $\varkappa = \phi_{k-1}\ldots\phi_1\phi_0$ is an involutive correlation of $\mathfrak{M}_0 = \langle \mathrm{S}_0, \mathrm{L}_0, \mathrel{I}_0\rangle$ then $\circledast_{i\in I}(\mathfrak{M}_i, \phi_i) \cong k \circledast_\varkappa \mathfrak{M}_0$.*

Proof. Let $\circledast_{i\in I}(\mathfrak{M}_i, \phi_i) = \langle X, H, \mathrel{I}\rangle$ and

$$\alpha_i := \phi_0^{-1}\ldots\phi_{i-2}^{-1}\phi_{i-1}^{-1}, \quad \beta_i := \phi_{k-1}\ldots\phi_{i+1}\phi_i.$$

We set the following bijective map δ:

$$\delta((i,x)) = \begin{cases} (i, \alpha_i(x)) & \text{for } i = 2t \\ (i, \beta_i(x)) & \text{for } i = 2t+1 \end{cases}, \quad (i,x) \in X \cup H.$$

Let $(i,a) \mathrel{I} [j,m]$ for some $(i,a) \in X$, $[j,m] \in H$. We claim that $\delta((i,a)) \mathrel{I} \delta((j,m))$ in sense of the relation $\mathrel{I}$ introduced by (1). The only not evident case is that with $i = j+1$ and $a = \phi''_j(m) = \phi''_{i-1}(m)$. Let i be even. Then we get $\delta((i,a)) = (i, \alpha_i(a)) = (i, \alpha_{i-1}\phi^{-1}_{i-1}(a)) = (i, \alpha_{i-1}(m))$, and $\delta((j,m)) = \delta((i-1,m)) = (i-1, \beta_{i-1}(m))$. Let $\varkappa_0 = \phi_{k-1}\ldots\phi_1\phi_0$ be an involutive correlation of $\mathfrak{M}_0$. Note, that formulas $\varkappa\,(\beta_{i-1}(m)) = \alpha_{i-1}(m)$ and $\varkappa^2\,(m) = m$ are equivalent, and they hold since $\varkappa^2 = id$. In order to get the claim for odd i we apply similar reasoning.

Hence, the map δ is an isomorphism between the structures $\circledast_{i\in I}(\mathfrak{M}_i, \phi_i)$ and $k \circledast_\varkappa \mathfrak{M}_0$. □

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